Determination of the Sp(2d,R) generator matrix elements through a boson mapping

1984 J. Phys. A: Math. Gen. 17 L405

(http://iopscience.iop.org/0305-4470/17/8/002)

J. Phys. A: Math. Gen. 17 (1984) L405–L409. Printed in Great Britain

LETTER TO THE EDITOR

Determination of the Sp(2*d*, *R*) generator matrix elements through a boson mapping

J Deenen and C Quesne†

Physique Théorique et Mathématique CP 229, Université Libre de Bruxelles, Bd du Triomphe, B 1050 Brussels, Belgium

Received 19 March 1984

Abstract. For the discrete series irreducible representations $\langle(\lambda + n/2)^d\rangle$ of Sp(2*d*, *R*), the determination of the Sp(2*d*, *R*) generator matrix elements in an $\mathrm{Sp}(2d, R) \supset \mathrm{U}(d)$ basis is reduced to the much simpler calculation of boson operator matrix elements between $\mathrm{U}(\nu) \supset \mathrm{U}(d)$ boson states, where $\nu = d(d+1)/2$. The key of this reduction is the previously derived Holstein–Primakoff boson representation of the Sp(2*d*, *R*) generators. As an illustration, the case of Sp(6, *R*) is worked out in detail.

During the last few years, the real symplectic group Sp(2*d*, *R*) has played an ever-increasing role in physical applications. Its importance is largely due to the fact that it is the main component of the *d*-dimensional harmonic oscillator dynamical group (Moshinsky and Quesne 1971, Wybourne 1974). Interest for the group Sp(6, *R*) has also arisen from its appearance as the dynamical group of a microscopic nuclear collective model (Rosensteel and Rowe 1980, Vasilevskii *et al* 1980, Castaños *et al* 1982, Deenen and Quesne 1982). The Sp(2*d*, *R*) irreducible representations (irreps) encountered in all these physical applications are positive discrete series (Rosensteel and Rowe 1977, Klimyk 1983), characterised by their lowest weight $\langle\lambda_d + n/2, \ldots, \lambda_1 + n/2\rangle$, where $[\lambda_1\lambda_2 \ldots \lambda_d]$ is a partition, and n is an integer greater than or equal to $2d$.

For practical purposes, it is important to determine for such irreps the matrix elements of the Sp(2*d*, *R*) generators in an $\mathrm{Sp}(2d, R) \supset \mathrm{U}(d)$ basis. In the case of Sp(6, *R*), some methods are available to calculate these matrix elements; they are based upon either recursion relations (Rosensteel 1980), or generating function techniques (Vasilevskii *et al* 1980). It was however recently realised that for some irreps simple analytic formulae can be obtained, thereby considerably easing the matrix element calculation for such irreps (Castaños *et al* 1984).

The purpose of the present letter is to show that for the irreps $\langle(\lambda + n/2)^d\rangle$, corresponding to $\lambda_1 = \lambda_2 = \ldots = \lambda_d = \lambda$, the determination of the Sp(2*d*, *R*) generator matrix elements in an $\mathrm{Sp}(2d, R) \supset \mathrm{U}(d)$ basis can be reduced to the much simpler calculation of boson operator matrix elements between boson states classified according to irreps of the group chain $\mathrm{U}(\nu) \supset \mathrm{U}(d)$, where $\nu = d(d+1)/2$. The key of this reduction is the Holstein–Primakoff (1940) boson representation of the Sp(2*d*, *R*) generators, recently derived by the present authors (Deenen and Quesne 1982).

As is well known (Moshinsky and Quesne 1971), the Sp(2*d*, *R*) generators can be realised in terms of dn boson creation operators η_{is}, $i = 1, \ldots, d$, $s = 1, \ldots, n$, and the

† Maître de recherches FNRS.

corresponding annihilation operators $\xi_{is} = (\eta_{is})^+$, as $D_{ij}^+ = D_{ji}^+ = \Sigma_{s=1}^n \eta_{is}\eta_{js}$, $D_{ij} = D_{ji} = \Sigma_{s=1}^n \xi_{is}\xi_{js}$, and $E_{ij} = \Sigma_{s=1}^n \eta_{is}\xi_{js} + (n/2)\delta_{ij}$, where $i, j = 1, \ldots, d$. The operators E_{ij} generate the maximal compact subgroup U(d) of Sp($2d$, R), and their commutators with the operators D_{kl}^+ are given by

$$[E_{ij}, D_{kl}^+] = \delta_{jk}D_{il}^+ + \delta_{jl}D_{ik}^+. \tag{1}$$

A basis of the irrep $\langle(\lambda + n/2)^d\rangle$ representation space can be built from the lowest weight state $|\lambda\rangle$ by applying polynomial functions in the D_{ij}^+ generators. By definition, $|\lambda\rangle$ satisfies the following equations

$$D_{ij}|\lambda\rangle = 0, \qquad E_{ij}|\lambda\rangle = \delta_{ij}(\lambda + n/2)|\lambda\rangle, \tag{2}$$

from which it results that it is the single state of the U(d) irrep $[(\lambda + n/2)^d]$ representation space. Polynomial functions $P_{[h_1 \ldots h_d](h)}(D_{ij}^+)$, characterised by a definite U(d) irrep $[h_1 \ldots h_d]$, and a row index (h)†, can be easily constructed for any even integer values of $h_1, \ldots, h_d$ (Deenen and Quesne 1982). The states

$$|\lambda; [h_1 \ldots h_d](h)\rangle = A_{h_1 \ldots h_d} P_{[h_1 \ldots h_d](h)}(D_{ij}^+)|\lambda\rangle \tag{3}$$

are then basis states of the U(d) irrep $[h_1 + \lambda + n/2, \ldots, h_d + \lambda + n/2]$ representation space. Since this irrep appears with multiplicity one in the reduction of $\langle(\lambda + n/2)^d\rangle$ into U(d) irreps, the states (3) corresponding to all positive even integer values of $h_1, \ldots, h_d$ acutally form an Sp($2d$, R) $\supset$ U(d) basis. In equation (3), $A_{h_1 \ldots h_d}$ is the normalisation coefficient of the highest weight state. Its explicit value (chosen as real) could in principle be determined from the commutaion relations of the Sp($2d$, R) generators. We shall however use a much simpler procedure to calculate it.

Let us now consider $\nu = d(d+1)/2$ independent boson creation and annihilation operators $a_{ij}^+ = a_{ji}^+$ and $a_{ij} = a_{ji}$, $i, j = 1, \ldots, d$, or the associated non-normalised operators $\bar{a}_{ij}^+ = \bar{a}_{ji}^+ = (1 + \delta_{ij})^{1/2} a_{ij}^+$ and $\bar{a}_{ij} = \bar{a}_{ji} = (1 + \delta_{ij})^{1/2} a_{ij}$, whose commutation relations are given by (Deenen and Quesne 1982)

$$[\bar{a}_{ij}, \bar{a}_{kl}^+] = \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}. \tag{4}$$

The two sets of operators $\mathfrak{E}_{ij,kl} = a_{ij}^+ a_{kl}$ and $\mathfrak{E}_{ij} = \Sigma_k \bar{a}_{ik}^+ \bar{a}_{kj}$ are respectively the generators of a U(ν) group and of its U(d) subgroup.

N boson states belong to an irrep $[N]$ of U(ν), and can be further characterised by a given irrep $[h_1 \ldots h_d]$ of U(d), and a row index (h). Only the irreps $[h_1 \ldots h_d]$, where $h_1, \ldots, h_d$ are even integers and satisfy the relation $\Sigma_i h_i = 2N$, are contained in $[N]$, and their multiplicity is equal to one. The commutation relation

$$[\mathfrak{E}_{ij}, \bar{a}_{kl}^+] = \delta_{jk}\bar{a}_{il}^+ + \delta_{jl}\bar{a}_{ik}^+ \tag{5}$$

shows that the boson operators $\bar{a}_{kl}^+$ have the same transformation properties with respect to the U(d) subgroup of U(ν) as the generators D_{kl}^+ with respect to the U(d) subgroup of Sp($2d$, R). The N boson states may therefore be written as

$$|[h_1 \ldots h_d](h)\rangle = B_{h_1 \ldots h_d} P_{[h_1 \ldots h_d](h)}(\bar{a}_{ij}^+)|0\rangle, \tag{6}$$

where $|0\rangle$ is the boson vacuum state, and $P_{[h_1 \ldots h_d](h)}(\bar{a}_{ij}^+)$ is the same polynomial function as $P_{[h_1 \ldots h_d](h)}(D_{ij}^+)$, but with D_{ij}^+ replaced by $\bar{a}_{ij}^+$. The normalisation coefficient $B_{h_1 \ldots h_d}$ in

† Since we only deal with reduced matrix elements of operators, it does not matter whether (h) denotes a Gel'fand pattern associated with the canonical chain of U(d) (Gel'fand and Tseitlin 1950, Baird and Biedenharn 1963), or another set of quantum numbers corresponding to a non-canonical chain.

equation (6) is however different from that of equation (3), because its value (chosen as real) results from the commutation relations (4) of the boson operators.

It is obvious that there is a one-to-one correspondence between the $Sp(2d, R) \supset U(d)$ basis states (3) and the $U(\nu) \supset U(d)$ boson states (6). The mapping of the former onto the latter, obtained by equating them, leads to a boson representation of the $Sp(2d, R) \supset U(d)$ basis states and of the $Sp(2d, R)$ generators (Deenen and Quesne 1982), which is of Holstein–Primakoff type (1940). This mapping may be written as

$$P_{[h_1\ldots h_d](h)}(D^+_{ij})|\lambda\rangle = u_{h_1\ldots h_d} P_{[h_1\ldots h_d](h)}(\bar{a}^+_{ij})|0), \tag{7}$$

where $u_{h_1\ldots h_d}$ is given by

$$u_{h_1\ldots h_d} = [A_{h_1\ldots h_d}]^{-1} B_{h_1\ldots h_d}. \tag{8}$$

Let us introduce an operator U, defined by its action on the boson states as follows

$$U|[h_1 \ldots h_d](h)) = u_{h_1\ldots h_d}|[h_1 \ldots h_d](h)). \tag{9}$$

This operator is clearly Hermitian and scalar under $U(d)$, but it is not unitary. In terms of it, equation (7) becomes

$$P_{[h_1\ldots h_d](h)}(D^+_{ij})|\lambda\rangle = U P_{[h_1\ldots h_d](h)}(\bar{a}^+_{ij})|0), \tag{10}$$

and the $Sp(2d, R)$ generators can be expressed as

$$D^+_{ij} = U\bar{a}^+_{ij}U^{-1}, \qquad D_{ij} = U^{-1}\bar{a}_{ij}U, \qquad E_{ij} = \mathfrak{E}_{ij} + (\lambda + n/2)\delta_{ij}. \tag{11a, b, c}$$

Equations (11a) and (11c) are a direct consequence of equation (10), while (11b) is the Hermitian conjugate of (11a).

The operator U, or more precisely its eigenvalues $u_{h_1\ldots h_d}$, can be directly determined from equation (11) and the commutation relations of the $Sp(2d, R)$ generators. It was shown elsewhere (Deenen and Quesne 1982) that the equations to be solved can be written as

$$U^2\bar{a}^+_{ij}U^{-2} = \sum_k [\mathfrak{E}_{ik} + (2\lambda + n - d - 1)\delta_{ik}]\bar{a}^+_{kj}, \tag{12}$$

and that their solution is given by

$$u_{h_1\ldots h_d} = \left(\prod_{i=1}^{d} \frac{(h_i + 2\lambda + n - i - 1)!!}{(2\lambda + n - i - 1)!!}\right)^{1/2}, \tag{13}$$

when choosing $u_{0\ldots 0} = 1$, and $u_{h_1\ldots h_d}$ positive.

It is now straightforward to prove the announced relation between matrix elements of operators in the $Sp(2d, R) \supset U(d)$ and $U(\nu) \supset U(d)$ basis. The only $Sp(2d, R)$ generators, whose matrix elements are required, are the D^+_{ij} operators. The matrix elements of D_{ij} can indeed be determined from them by using the Hermiticity properties of the operators, while those of E_{ij} are well known since the E_{ij} are the generators of a $U(d)$ group (Gel'fand and Tseitlin 1950, Baird and Biedenharn 1963). From equations (1) and (5), it is obvious that the operators D^+_{ij} and $\bar{a}^+_{ij}$ are the components of $[20\ldots 0]$ irreducible tensors with respect to the $U(d)$ subgroups of $Sp(2d, R)$ and $U(\nu)$ respectively. Such irreducible tensors may only induce transitions from an irrep

$[h_1 \ldots h_i \ldots h_d]$ to the irreps $[h_1 \ldots h_i+2 \ldots h_d]$, where $i=1, \ldots, d$, and no multiplicity index is required in their matrix elements. From equation (11*a*), the ratio of the reduced matrix elements of D^+_{ij} and $\bar{a}^+_{ij}$ is simply given by $u_{h_1 \ldots h_i+2 \ldots h_d}/u_{h_1 \ldots h_i \ldots h_d}$, i.e.,

$$\langle \lambda ; [h_1 \ldots h_i+2 \ldots h_d] \| D^+ \| \lambda ; [h_1 \ldots h_i \ldots h_d] \rangle = \left(\prod_{i=1}^{d} (h_i + 2\lambda + n - i + 1) \right)^{1/2} \langle [h_1 \ldots h_i+2 \ldots h_d] \| \bar{a}^+ \| [h_1 \ldots h_i \ldots h_d] \rangle. \tag{14}$$

As an additional point, we also note from equations (8) and (13) that the normalisation coefficient $A_{h_1 \ldots h_d}$ of the $\mathrm{Sp}(2d, R) \supset \mathrm{U}(d)$ basis states can be determined from the normalisation coefficient $B_{h_1 \ldots h_d}$ of the $\mathrm{U}(\nu) \supset \mathrm{U}(d)$ ones as follows

$$A_{h_1 \ldots h_d} = \left(\prod_{i=1}^{d} \frac{(2\lambda + n - i - 1)!!}{(h_i + 2\lambda + n - i - 1)!!} \right)^{1/2} B_{h_1 \ldots h_d}. \tag{15}$$

To illustrate our procedure, let us consider the case of the Sp(6, *R*) generators. In the microscopic nuclear collective model mentioned in the introduction, the irreps $\langle (\lambda + n/2)^3 \rangle$ of Sp(6, *R*) are encountered in closed-shell nuclei. For such irreps, equation (14) establishes a relation between the matrix elements of the Sp(6, *R*) generators D^+_{ij} in an $\mathrm{Sp}(6, R) \supset \mathrm{U}(3)$ basis and those of boson operators between boson states classified according to $\mathrm{U}(6) \supset \mathrm{U}(3)$. Analytic formulae being available for the latter (Quesne 1981), we obtain from equation (14) the following results

$$\begin{aligned}
&\langle \lambda ; [h_1+2, h_2, h_3] \| D^+ \| \lambda ; [h_1 h_2 h_3] \rangle \\
&\quad = \left(\frac{(h_1+4)(h_1+2\lambda+n)(h_1-h_2+2)(h_1-h_3+3)}{(h_1-h_2+3)(h_1-h_3+4)} \right)^{1/2}, \\
&\langle \lambda ; [h_1, h_2+2, h_3] \| D^+ \| \lambda ; [h_1 h_2 h_3] \rangle \\
&\quad = \left(\frac{(h_2+3)(h_2+2\lambda+n-1)(h_1-h_2)(h_2-h_3+2)}{(h_1-h_2-1)(h_2-h_3+3)} \right)^{1/2}, \\
&\langle \lambda ; [h_1, h_2, h_3+2] \| D^+ \| \lambda ; [h_1 h_2 h_3] \rangle \\
&\quad = \left(\frac{(h_3+2)(h_3+2\lambda+n-2)(h_1-h_3+1)(h_2-h_3)}{(h_1-h_3)(h_2-h_3-1)} \right)^{1/2}.
\end{aligned} \tag{16}$$

Proceeding in the same way for the normalisation coefficient of the $\mathrm{Sp}(6, R) \supset \mathrm{U}(3)$ basis states, from the corresponding result for the $\mathrm{U}(6) \supset \mathrm{U}(3)$ boson states (Quesne 1981) and equation (15) we obtain the relation

$$\begin{aligned}
A_{h_1 h_2 h_3} = {} & [(\tfrac{1}{2}h_2)!(\tfrac{1}{2}(h_1-h_3)+1)!(h_1-h_2+1)!!(h_2-h_3+1)!!(2\lambda+n-2)!!(2\lambda+n-3)!! \\
& \times (2\lambda+n-4)!!]^{1/2} [(\tfrac{1}{2}h_1+1)!(h_2+1)!(\tfrac{1}{2}(h_2-h_3))!(h_1-h_2)!! \\
& \times (h_1-h_3+1)!! h_3!!(h_1+2\lambda+n-2)!! \\
& \times (h_2+2\lambda+n-3)!!(h_3+n+2\lambda-4)!!]^{-1/2}.
\end{aligned} \tag{17}$$

Both equations (16) and (17) agree with the results obtained by Castaños *et al* (1984) but after some rather lengthy calculations.

References

Baird G E and Biedenharn L C 1963 *J. Math. Phys.* **4** 1449
Castaños O, Chacon E, Frank A, Hess P and Moshinsky M 1982 *J. Math. Phys.* **23** 2537
Castaños O, Chacon E and Moshinsky M 1984 *J. Math. Phys.* In press
Deenen J and Quesne C 1982 *J. Math. Phys.* **23** 878, 2004
Gel'fand I M and Tseitlin M L 1950 *Dokl. Akad. Nauk* **71** 825
Holstein T and Primakoff H 1940 *Phys. Rev.* **58** 1098
Klimyk A U 1983 *J. Math. Phys.* **24** 224
Moshinsky M and Quesne C 1971 *J. Math. Phys.* **12** 1772
Quesne C 1981 *J. Math. Phys.* **22** 1482
Rosensteel G 1980 *J. Math. Phys.* **21** 924
Rosensteel G and Rowe D J 1977 *Int. J. Theor. Phys.* **16** 63
—— 1980 *Ann. Phys., NY* **126** 343
Vasilevskii V S, Smirnov Yu F and Filippov G F 1980 *Sov. J. Nucl. Phys.* **32** 510
Wybourne B G 1974 *Classical Groups for Physicists* (New York: Wiley)